Ancient History of CFT

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- These are exciting times in Mathematical Physics. Long-awaited exactly solvable 4D Conformal Field Theory is found and is being investigated. It may be worth recalling at this moment how the notions of anomalous dimensions and conformal invariance were born in the 60-ties and 70-ties.

- There is remarkable analogy between QFT and Statistical Physics, allowing for cross-fertilization. This analogy was foreseen by great Julian Schwinger, who noted that inverse temperature $\beta = 1/T$ in Statistical Physics is equivalent to imaginary time in QFT
- $Z = \mathrm{Trace}(\exp(-\beta H))$ in statistics and
- $Z = \mathrm{Trace}(\exp(-\acute{\imath}\, t H))$ in quantum theory.

- The missing link in the 60-ties was a realization that this “statistical” imaginary time is the same as Minkovski imaginary time of special relativity. This is the same observation that drove Steven Hawking to interpret the imaginary time of the black hole as inverse temperature. Good ideas are so scarce they come back again and again in different disguise.

- The QFT was pronounced dead in late 50-ties by Heisenberg and Landau. As Landau put it “The Lagrangean Field Theory is dead and should be buried, with all the proper honors of course”. The Landau’s motivation was the “zero charge”, which, indeed, indicated inconsistency of all known field theories except YM, which was in its infancy and could not yet stand and defend itself.

- Heisenberg’s motivation was even more ambitious. This is an example how one great leader can block the way to the whole army by falling down at a narrow pass. He dared to go one step further from his celebrated uncertainty principle and declare that Physics must only study observable quantities. His own approach was to study so called S-Matrix – collection of transition amplitudes between various observable in- and out- states.

- Pretty much like medieval Scholastic Magisters were extremely inventive in defending the Church Dogmas and blocking the way to experimental science, some great minds in the sixties developed the S-Matrix dogma with great perfection and skill before it was buried down in the seventies after discovery of quarks and asymptotic freedom.

- It turned out – quite unfortunately for Physics – that one could deduce a lot about S-Matrix on purely phenomenological grounds without ever asking heretical questions “what is inside”. One could not, of course, even attempt to compute proton mass or its magnetic moment or explain anything about properties of so-called resonances – short living subatomic particles.

- In a way, this reactionary idea was a truly revolutionary one. For the first time since Galileo the quest for the structure of matter was stopped on philosophical grounds. There is nothing inside – total nuclear democracy! Everything consists of everything else – do not ask whether there was the rabbit inside the hat – you are only allowed to compute how far it will jump and in what direction.

- It is a bitter irony of History, that such a restriction on a free thought was imposed by a German Scientist and so widely accepted in Russia in the second half of 20th century. My Physics teachers Gribov and Okun were respected as liberals and free thinkers, followers of great Landau, but still they would not even talk to me about Yang-Mills Theory because it was “unobservable”.

- For the whole two years 1964 to 1966 JETP refused to publish our work with Sasha Polyakov “Spontaneous Symmetry Breaking of Strong Interaction and Absence of Massless Particles” where we (correctly!) argued that vector mesons of the Yang-Mills Theory must acquire mass by absorbing zero mass Goldstone particles. We were stomped to the ground at every seminar we tried to present this work at.

- As for the Lagrangean Field Theory, so respectfully buried by Heisenberg and Landau, my good friend Sasha Zamolodchikov (another Sasha from Landau Institute) summarized it like that: “They buried the Lagrangean Field Theory, but forgot to drive the stake through the heart”.

- The ideas which eventually led to Conformal Field Theory were first expressed by Landau in unpublished work just before the 1962 automobile accident. According to his graduate student at that time, Sasha Patashinsky, Landau studied 3D field theory of fluctuations in Critical Phenomena (liquid Helium at l point, ferromagnetic at Curie point etc.).

- He observed that unlike all known local 4D quantum field theories at that time (not to mention Yang-Mills which was not yet quantized), the 3D field theory was not killed by zero charge problem. Nobody knows now how much analogy with Euclidean Relativistic Theory he observed, as the latter theory was a heresy at that time, and he was the Great Inquisitor Himself.

- What he did observe was the k$^{3/2}$ law which was a first attempt to get anomalous dimensions in field theory. This law came about if one neglected the bare propagator k^2 in the massless lj^4 field theory in 3 dimensions. All the self energy graphs balanced their powers in Dyson equation
- G = -1/S ; S = l^2 òòGGG + …

- However, there was a problem, which he was either unaware of or did not have time to solve before the accident. When the $k^{3/2}$ propagator was inserted back in the Feynman graphs of the lj^4 theory, the vertex part diagrams started to diverge logarithmically.
- It fascinates me to imagine that he was thinking about these problems when I integrated rational function for him at my first Landau exam in 1961.

- Valery Pokrovsky and Sasha Patashinsky took it from Landau in mid-60-ties after his accident. They wrote very interesting paper summarizing the $k^{3/2}$ idea, noting the logarithmic divergence problem and trying to sweep it under the rug. They expressed the hope that the divergent terms will conspire to cancel among themselves.

- I had a privilege to be the next player in this game. I was inspired by the $k^{3/2}$ idea and tried to apply it to another long range theory, namely Gribov’s Reggeon Field theory. This was a theory of 2+1 non-relativistic particles with spectrum $E = k^2$, interacting via $\imath g j^3$ Field Theory with purely imaginary coupling constant $\imath g$.

- The notion of scale invariance applied to this theory, but I observed that there was another possible critical index, -- namely that of dynamic spectrum $E = k^{\mu}$.The power counting in spirit of Landau, neglecting the bare propagator, established connection between μ and the power dimension of the propagator.

- I came to Gribov with these ideas and was fiercely criticized by him in the best traditions of Landau school. However, I did not crawl away, and kept coming with answers to his objections, so we started working together on this scale invariant theory.
- We had two problems: the logarithmic divergence, and the free critical index, without any apparent idea how to compute it.

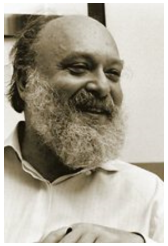
- As for logarithmic divergence, we soon observed that the logarithms add up into exponentials and transform to powers. But if we assumed powerlike vertex parts in addition to the powerlike propagarors, all the initial parameters dropped from these “bootstrap” equations, which sounded like nonsense at that time.
- These equations were homogeneous, which meant that there was always a trivial zero solution, and it was a mystery how to obtain another solution.

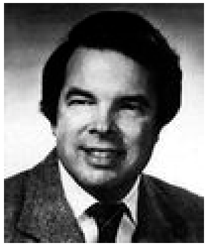
- I vividly remember the breakthrough in this problem. I woke up one morning in winter of 1966-67 in Leningrad. I called Gribov, apparently waking him up, and I repeated to him the first thing which came to my mind that morning: “We have a free dimensionless parameter μ. Maybe it must be tuned to make homogeneous equation have non-zero solution, like it is done in the linear eigenvalue equations?” There was a long silence on the other end of the line and then Gribov slowly said:” You know, it might work…”.

- This is the first time to the best of my knowledge that the idea of anomalous dimensions as eigenvalues of certain field theory equations was ever expressed. Afterwards, in my PHD thesis I realized that the anomalous dimensions are related to RG equations for the running coupling constant, which constant tends to the the root of beta function for self-consistency.

- Later, Ken Wilson elaborated this idea to total clarity, and found practical computation method by his RG approach to Critical Phenomena. The Bootstrap equations corresponded to fixed point of Renormalization group, and eigenvalue equations for critical indexes came about as spectrum of linear perturbations around the fixed point.

- Sasha Polyakov and I applied the idea of anomalous dimensions to the Phase Transition Theory the same year while Gribov and I were slowly preparing the Strong Coupling Reggeon Theory for publication. This was 1967. I must say, that in the retrospect, reading these old papers, I am proud of the idea of anomalous dimensions, and consider it my most important personal achievement.





- To be fair, I must also add, that Sasha Polyakov understood the Phase transition problem much deeper than me, and went much further in developing the idea of anomalous dimensions. I especially admire his idea of correlation joining, equivalent to Operator Product Expansion, later and independently developed by Kadanoff and Wilson.

- Conformal Field Theory was the next step of development of the idea of anomalous dimensions, based on remarkable observation that one-dimensional scale invariance in local Euclidean Field Theory necessarily leads to a wider symmetry, with 15 parameters in our four dimensions (including translations and rotations). The generators of conformal symmetry are related to various conserved currents

- $K_\nu(X) = X_\mu \Theta_{\mu\nu}(X)$, $\partial_\nu K_\nu(X) = 0$;
- for dilatations
- $C_{\mu\nu}(X) = (X^2 \delta_{\mu\lambda} - 2 X_\mu X_\lambda)\Theta_{\lambda\nu}(X)$, $\partial_\nu C_{\mu\nu}(X) = 0$;
- for special conformal transformations, etc.
- Here the conserved symmetric stress-energy tensor $\Theta_{\mu\nu}$ is in addition traceless in case there is scale invariance (no massive fields), which leads to conservation of both of these currents simultaneously.

- Sasha Polyakov read about it in some obscure paper, where it was applied to a free massless theory, and immediately combined it with idea of anomalous dimensions. He wrote a short but seminal note in JeTP letters on the subject, elaborating the idea for the critical phenomena. In particular, he was the first to write down famous formula $(x_{12}\, x_{23}\, x_{31})^{-D}$ for the 3-point function with relative distances x_{ij} and observe orthogonality of 2-point functions with different dimensions.

- At the same time, I instantly realized that in CFT with anomalous dimensions, where the 3-point functions are uniquely fixed by conformal symmetry, my old bootstrap equations reduce to transcendental equations for anomalous dimensions. Every Feynman diagram for the 3-point function with conformal propagators and conformal vertexes inside, produces the same 3-point function up to normalization constant calculable by taking the limit when one point goes to spatial infinity.

- I also noted, that the expansion of 3-point function at approaching points, analytically continued in Minkovski space, leads to conformal Operator Product Expansion, with calculable coefficients in front of derivatives of operators. This paper was submitted in Physics Letters, but was delayed by a year in the hands of dishonest Editor and referees, who stole my Conformal OPE and held my publication until their own paper was finished. I remember how mad I was when I found it out. I must say to the credit of the Editor that he met me later in CERN and apologized.

- Sasha and I were very excited by grandiose perspectives we saw in CFT (it eventually became one of the basic ingredients of the modern Mathematical Physics), so we kept trying to discuss it with our Soviet colleagues (western colleagues appreciated it immediately).

- It happened so, that there was an International Conference in Dubna, the main topic of which was so-called scale symmetry, promoted with great fanfare by the Bogoliubov School. This scale symmetry was mostly a political slogan, good for dissertations and career moves but not for any practical applications in a world of Physics.
- After the Plenary Session devoted to the Scale Symmetry, one of the Western Physicists asked the speaker: “What is the difference between Scale Symmetry and Conformal Symmetry?” Apparently, the rumors about new symmetry were already spread, so this was what KGB used to call “provocative question”.

- The speaker hesitated, but the Chairman of the Session, great mathematician N.N. Bogoliubov took the microphone and said literally the following: “There is no mathematical difference, but when some young people want to use a fancy word they call it Conformal Symmetry”.
- Obviously, his ignorant lieutenants misinformed him, and he did not bother to look up for himself what was the Conformal Symmetry.

- I could not stand it any longer – I raised the hand to give everybody brief introduction to Conformal Symmetry. Vigilant Organizers of the Conference ignored my raised hand, the break was quickly announced, so that my indignant yell: “15 parameters!” went apparently unnoticed.
- By the way, somebody told me recently that he heard that yell and wondered for years what that could mean, until he learned conformal symmetry.

- Here is an interesting part – I came home and said to my father: ”Look, what a fool N.N is really is ” – then I told him the story of 15 parameters. My father laughed with me – surely he knew what Conformal Symmetry was about – then he said something remarkable: “You know, Sasha, there are two kinds of wisdom. The first kind helps you to **say** smart things. But the second kind helps you to **do** smart things. N.N used to have great wisdom of the first kind, but later he switched to the second kind. Do you think he cares about parameters of Conformal Group? He is involved in Big Science, where political truth is more important than scientific truth. You would do yourself some good by borrowing the second kind of wisdom from N.N.”

- After the discovery of Asymptotic Freedom it became immediately clear, that CFT in 4 dimensions can exist – one should simply arrange so that the beta function has the root close to zero coupling constant.
- The leading term in the beta function was proportional to $(11\, N_c - 2\, N_f)$, so that at large numbers this leading term could be kept finite. This resulted in calculable fixed point with coupling constant proportional to this term.

- At the same time the SUSY was discovered, which eventually led to discovery of N=4 SYM CFT with exactly vanishing beta function in 4 dimensions. My dream of 4D CFT finally came true, but not quite the way I expected. This theory is perturbative, so that one actually sums up Feynman graphs like we did in the 60-ties.
- In the limit of large number of colors this theory is exactly solvable, and is show to be equivalent to a certain “string theory” in 5D curved space without string tension.

- What can be done with this remarkable exactly solvable theory? Our ultimate dream is to solve Asymptotically Free QCD at large N_c .The hopes for solving Large N_c QCD come from its simple analytic structure: it is the theory of infinite number of free particles, corresponding to meromorphic 2-point functions in momentum space.

- In N=4 SYM theory, like in any CFT, the 2-point functions are rather powers of momentum, corresponding to all the poles condensing to zero. Let us assume now that conformal symmetry is broken by some soft mechanism in the infrared region (like QCD, of like N=4 SUSY YM with SUSY broken by mass terms for scalar fields). In that case the spectrum must be discrete, but in the UV limit the CFT must be recovered.

- 30 years ago, in 1977, I wrote a paper on this subject, which was recently brought back to life, because my old solution for the mass spectrum happened to exactly coincide with results coming from the 5D string theory regularized along the 5th dimension.
- Unfortunately, only first few pages of my old paper were revived, the rest was kept in the dust of Ancient history. This led to certain mistakes, which I would like to correct.

- Namely, the soft perturbations of the Bessel roots spectrum of massless CFT with IR regularization by Pade method from my old paper, do not vanish, contrary to statements in the literature. These corrections were given by some multiple Bessel function integrals in my old paper, which I recently reduced to explicit simple analytic formulas. I presented these formulas at the Cocoyoc Symposium in Mexico in February 2007. The copies of my talk are available for those who may need them, just email me or ask for a reprint.